A STUDY OF THE EQUILIBRIUM EQUATIONS OF THIN ELASTIC SHELLS OF POSITIVE GAUSSIAN CURVATURE

(ISSLEDOVANIE URAVNENII RAVNOVESIIA TONKIKH UPRUGIKH OBOLOCHEK POLOZHITEL'NOI GAUSSOVOI KRIVIZNY)

PMM Vol.23, No.1, 1959, pp. 134-145

V. S. ZHGENTI
(Tbilisi)

(Received 15 March 1958)

This paper contains a study, from the point of view of equations with a small parameter associated with higher derivatives, of thin elastic shells of a general shape, defined on surfaces of positive Gaussian curvature and fixed on the contour.

1. The operator of the theory of thin elastic shells and the formulation of the basic theorem. We introduce on the middle surface S of the shell the orthogonal curvilinear coordinates x_1, x_2, where x_1 = const and x_2 = const are lines of curvature. Let us designate by G the region of change of parameters x_1, x_2 on the plane $x_1 + ix_2$, which corresponds to the surface S.

Let us assume that the boundary of the region G is a sufficiently smooth, closed, nonintersecting curve I.

Let the length of an elementary arch be

$$ds^2 = A_1^2(x_1, x_2)\, dx_1^2 + A_2^2(x_1, x_2)\, dx_2^2$$

We will assume that the functions A_1, A_2 are continuously differentiable to a sufficiently high order in $G + L$ and $A_1 > 0$, and $A_2 > 0$ in $G + L$.

Let the Gaussian curvature of the middle surface S be positive, i.e.

$$k_1 k_2 > 0$$

where k_1 and k_2 are the principal curvatures of the same surface.

Let e_1, e_2, e_3 be the unit vectors along tangents to the lines x_1, x_2 and to the normal to the middle surface S respectively.

We start with relations [1,2]

$$\varepsilon_{11}(\mathbf{u})=\frac{1}{A_1}\frac{\partial u_1}{\partial x_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2-k_1u_3$$

$$\varepsilon_{12}(\mathbf{u})=\varepsilon_{21}(\mathbf{u})=\frac{1}{2}\left\{\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\left(\frac{u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial x_1}\left(\frac{u_2}{A_1}\right)\right\} \tag{1.1}$$

$$\varepsilon_{22}(\mathbf{u})=\frac{1}{A_2}\frac{\partial u_2}{\partial x_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}u_1-k_2u_3$$

$$\varkappa_{11}(\mathbf{u})=-\frac{1}{A_1}\frac{\partial k_1}{\partial x_1}u_1-\frac{1}{A_2}\frac{\partial k_1}{\partial x_2}u_2-k_1^2u_3-\frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial x_2}\frac{\partial u_3}{\partial x_2}-\frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial x_1}\right)$$

$$\varkappa_{12}(\mathbf{u})=\varkappa_{21}(\mathbf{u})=\frac{1}{2}\left\{(k_2-k_1)\left[\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\left(\frac{u_1}{A_1}\right)-\frac{A_2}{A_1}\frac{\partial}{\partial x_1}\left(\frac{u_2}{A_2}\right)\right]-\right.$$

$$\left.-\frac{A_1}{A_2}\frac{\partial}{\partial x_2}\left(\frac{1}{A_1^2}\frac{\partial u_3}{\partial x_1}\right)-\frac{A_2}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_2^2}\frac{\partial u_3}{\partial x_2}\right)\right\} \tag{1.2}$$

$$\varkappa_{22}(\mathbf{u})=-\frac{1}{A_1}\frac{\partial k_2}{\partial x_1}-\frac{1}{A_2}\frac{\partial k_2}{\partial x_2}u_2-k_2^2u_3-\frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial x_1}\frac{\partial u_3}{\partial x_1}-\frac{1}{A_2}\frac{\partial}{\partial x_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial x_2}\right)$$

$$\left(\mathbf{u}=\sum_{i=1}^{3}u_i\mathbf{e}_i\right)$$

where **u** is the vector of small displacements of the points on the middle surface S.

The potential energy of deformation of a thin shell will be of the form [2] :

$$\frac{E}{2(1-\sigma^2)}\iint_G\left\{h\left[\varepsilon_{11}^2+2\sigma\varepsilon_{11}\varepsilon_{22}+\varepsilon_{22}^2+2(1-\sigma)\varepsilon_{12}^2\right]+\right.$$

$$\left.+\frac{h^3}{12}\left[\varkappa_{11}^2+2\sigma\varkappa_{11}\varkappa_{22}+\varkappa_{22}^2+2(1-\sigma)\varkappa_{12}^2\right]\right\}A_1A_2dx_1dx_2$$

where h is the thickness of the shell, E is Young's modulus and σ is Poisson's ratio.

The differential equations of equilibrium of the thin elastic shell obtained from the principle of the minimum potential energy will be written down in the form

$$h(\mathbf{Bu}+h^2\mathbf{Nu})=A_1A_2\mathbf{q} \tag{1.3}$$

where **q** is the external loading and

$$\mathbf{Bu}=\sum_{i=1}^{3}(B_i\mathbf{u})\mathbf{e}_i,\qquad \mathbf{Nu}=\sum_{i=1}^{3}(N_i\mathbf{u})\mathbf{e}_i$$

where

$$B_1\mathbf{u}=\frac{E}{1-\sigma^2}\left\{-\frac{\partial}{\partial x_1}\left[A_2(\varepsilon_{11}+\sigma\varepsilon_{22})\right]+(\sigma\varepsilon_{11}+\varepsilon_{22})\frac{\partial A_2}{\partial x_1}-\frac{1-\sigma}{A_2}\frac{\partial}{\partial x_2}(A_1^2\varepsilon_{12})\right\}$$

$$B_2\mathbf{u} = \frac{E}{1-\sigma^2}\left\{-\frac{\partial}{\partial x_2}[A_1(\sigma\varepsilon_{11}+\varepsilon_{22})] + (\varepsilon_{11}+\sigma\varepsilon_{22})\frac{\partial A_1}{\partial x_2} - \frac{1-\sigma}{A_2}\frac{\partial}{\partial x_1}(A_2{}^2\varepsilon_{12})\right\}$$

$$B_3\mathbf{u} = -\frac{E}{1-\sigma^2}[(k_1+\sigma k_2)\varepsilon_{11} + (\sigma k_1 + k_2)\varepsilon_{22}]A_1A_2$$

$$N_1\mathbf{u} = \frac{E}{12(1-\sigma^2)}\left\{-A_2\left[\frac{\partial k_1}{\partial x_1}(\varkappa_{11}+\sigma\varkappa_{22}) + \frac{\partial k_2}{\partial x_1}(\sigma\varkappa_{11}+\varkappa_{22})\right] - \right.$$
$$\left. - \frac{1-\sigma}{A_1}\frac{\partial}{\partial x_2}[A_1{}^2(k_2-k_1)\varkappa_{12}]\right\}$$

$$N_2\mathbf{u} = \frac{E}{12(1-\sigma^2)}\left\{-A_1\left[\frac{\partial k_1}{\partial x_2}(\varkappa_{11}+\sigma\varkappa_{22}) + \frac{\partial k_2}{\partial x_2}(\sigma k_{11}+\varkappa_{22})\right] + \right.$$
$$\left. + \frac{1-\sigma}{A_2}\frac{\partial}{\partial x_1}[A_2{}^2(k_2-k_1)\varkappa_{12}]\right\}$$

$$N_3\mathbf{u} = \frac{E}{12(1-\sigma)}\left\{-A_1A_2[k_1{}^2(\varkappa_{11}+\sigma\varkappa_{22})] + k_2{}^2(\sigma\varkappa_{11}+\varkappa_{22}) + \right.$$
$$+ \frac{\partial}{\partial x_2}\left[\frac{1}{A_2}\frac{\partial A_1}{\partial x_2}(\varkappa_{11}+\sigma\varkappa_{22})\right] + \frac{\partial}{\partial x_1}\left[\frac{1}{A_1}\frac{\partial A_2}{\partial x_1}(\sigma\varkappa_{11}+\varkappa_{22})\right] -$$
$$- \frac{\partial}{\partial x_1}\frac{1}{A_1}\frac{\partial}{\partial x_1}[A_2(\varkappa_{11}+\sigma\varkappa_{22})] - \frac{\partial}{\partial x_2}\frac{1}{A_2}\frac{\partial}{\partial x_2}[A_1(\sigma\varkappa_{11}+\varkappa_{22})] -$$
$$\left. -(1-\sigma)\frac{\partial}{\partial x_1}\frac{1}{A_1{}^2}\frac{\partial}{\partial x_2}(A_1{}^2\varkappa_{12}) - (1-\sigma)\frac{\partial}{\partial x_2}\frac{1}{A_2{}^2}\frac{\partial}{\partial x_1}(A_2{}^2\varkappa_{12})\right\}$$

Let the vector $\mathbf{q}$ be continuous and continuously differentiable to a sufficiently high order in $G + L$.

Assume further that

$$\mathbf{q} = \mathbf{q}_0(x_1, x_2) + \mathbf{q}_1(x_1, x_2; h) \qquad (1.4)$$

where $\mathbf{q}_0 \not\equiv 0$ does not depend on h and

$$\lim_{h\to 0}\iint_G |\mathbf{q}_1|^2\, dx_1 dx_2 = 0 \qquad (1.5)$$

We introduce the vector $\mathbf{U} = h\mathbf{u}$ which, by virtue of (1.3) and (1.4), satisfies the equation

$$\mathbf{BU} + h^2\mathbf{NU} = A_1A_2[\mathbf{q}_0(x_1, x_2) + \mathbf{q}_1(x_1, x_2, h)] \qquad (1.6)$$

Let us consider the following two problems.

Problem A. Let the vector $\mathbf{U} = U_1\mathbf{e}_1 + U_2\mathbf{e}_2 + U_3\mathbf{e}_3$ satisfy the equation (1.6) and the boundary condition

$$U_1 = U_2 = U_3 = \frac{\partial U_3}{\partial \nu} = 0 \quad \text{on } L \qquad (1.7)$$

where ν is the normal to the curve L.

Problem B. Let the vector $\mathbf{U}_0 = U_{10}\mathbf{e}_1 + U_{20}\mathbf{e}_2 + U_{30}\mathbf{e}_3$ satisfy the equation

$$\mathbf{B}\mathbf{U}_0 = A_1 A_2 \mathbf{q}_0(x_1, x_2) \tag{1.8}$$

and the boundary condition

$$U_{10} = U_{20} = 0 \quad \text{on } L \tag{1.9}$$

We note that the problems A and B are formulated correctly.

Basic Theorem. If $\mathbf{U}$ and $\mathbf{U}_0$ are solutions of problems A and B, then

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 \tag{1.10}$$

where the vector $\mathbf{U}_1 = U_{11}\mathbf{e}_1 + U_{21}\mathbf{e}_2 + U_{31}\mathbf{e}_3$ has the form:

$$\begin{aligned} U_{11} &= \{a_1 h^{1/2}\cos(gh^{-1/2}) + h\,[b_1\sin(gh^{-1/2}) + c_1\cos(gh^{-1/2}) + d_1]\}\, e^{-gh^{-1/2}} \\ U_{21} &= \{a_2 h^{1/2}\cos(gh^{-1/2}) + h\,[b_2\sin(gh^{-1/2}) + c_2\cos(gh^{-1/2}) + d_2]\}\, e^{-gh^{-1/2}} \quad (1.11) \\ U_{31} &= \{a_3\,[\sin(gh^{-1/2}) + \cos(gh^{-1/2})] + b_3 h^{1/2}\,[\sin(gh^{-1/2}) - \cos(gh^{-1/2}) + 1]\}\, e^{-gh^{-1/2}} \end{aligned}$$

whereby the function g, determined in a certain neighborhood Ω of the boundary L, becomes zero on L and is positive at points of the region G. The vector $\mathbf{U}_2$ depends on h in such a fashion that

$$\lim_{h\to 0}\iint_G |\mathbf{U}_2|^2\, dx_1 dx_2 = 0 \tag{1.12}$$

2. Basic inequality. We consider the equation

$$\mathbf{B}\mathbf{v} + h^2\mathbf{N}\mathbf{v} = \mathbf{Q} \tag{2.1}$$

Let us now prove the following lemma.

Lemma. If the vector

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$$

satisfies the equation (2.1) and the boundary condition

$$v_1 = v_2 = v_3 = \frac{\partial v_3}{\partial \nu} = 0 \quad \text{on } L \tag{2.2}$$

then

$$\iint_G |\mathbf{v}|^2\, dx_1 dx_2 \leqslant \gamma \iint_G |\mathbf{Q}|^2\, dx_1 dx_2 \tag{2.3}$$

where γ is a positive constant number not dependent on h, $\mathbf{v}$ and $\mathbf{Q}$.

Inequality (2.3) will be called the basic inequality.

Proof. We introduce the notation

$$(\mathbf{v}_1, \mathbf{v}_2) = \iint\limits_G \mathbf{v}_1 \mathbf{v}_2 \, dx_1 dx_2$$

Applying the boundary conditions (2.2) and the inequality $a^2 + b^2 + 2\sigma ab \geqslant (1 - \sigma)(a^2 + b^2)$ we easily obtain

$$(\mathbf{Bv}, \mathbf{v}) \geqslant \frac{EA_0}{1+\sigma} \iint\limits_G \sum_{i,j=1}^{2} \varepsilon_{ij}^2 (\mathbf{v}) \, dx_1 dx_2 \tag{2.4}$$

where

$$A_1 A_2 \geqslant A_0 > 0 \quad \text{in } G + L$$

The following equation holds:

$$\sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{w}) A_1 A_2 =$$

$$= \sum_{i=1}^{3} v_i P_i (\mathbf{w}) + \frac{\partial}{\partial x_1} \{A_2 [v_1 \varepsilon_{11}(\mathbf{w}) + v_2 \varepsilon_{12}(\mathbf{w})]\} + \frac{\partial}{\partial x_2} \{A_1 [v_1 \varepsilon_{12}(\mathbf{w}) + v_2 \varepsilon_{22}(\mathbf{w})]\} \tag{2.5}$$

where

$$P_1 \mathbf{w} = -\frac{\partial}{\partial x_1} [A_2 \varepsilon_{11}(\mathbf{w})] + \frac{\partial A_2}{\partial x_1} \varepsilon_{22}(\mathbf{w}) - \frac{1}{A_1} \frac{\partial}{\partial x_2} [A_1^2 \varepsilon_{12}(\mathbf{w})]$$

$$P_2 \mathbf{w} = -\frac{\partial}{\partial x_2} [A_1 \varepsilon_{22}(\mathbf{w})] + \frac{\partial A_1}{\partial x_2} \varepsilon_{11}(\mathbf{w}) - \frac{1}{A_2} \frac{\partial}{\partial x_1} [A_2^2 \varepsilon_{12}(\mathbf{w})]$$

$$P_3 \mathbf{w} = -A_1 A_2 [k_1 \varepsilon_{11}(\mathbf{w}) + k_2 \varepsilon_{22}(\mathbf{w})]$$

Let us consider the system of differential equations

$$P_i \mathbf{w} = 0 \qquad (i = 1, 2, 3) \tag{2.6}$$

From the equation $P_3 \mathbf{w} = 0$ we obtain

$$w_3 = \frac{k_1}{k_1^2 + k_2^2} \left(\frac{1}{A_1} \frac{\partial w_1}{\partial x_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial x_2} w_2 \right) + \frac{k_2}{k_1^2 + k_2^2} \left(\frac{1}{A_2} \frac{\partial w_2}{\partial x_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial x_1} w_1 \right) \tag{2.7}$$

Introducing (2.7) into equations $P_1 \mathbf{w} = 0$, $P_2 \mathbf{w} = 0$, we obtain

$$\sum_{j=1}^{2} A_{ij} w_j = 0 \qquad \left(A_{ij} = \sum_{j_1 + j_2 \leqslant 2} f_{ij}^{j_1 j_2} \frac{\partial^{j_1 + j_2}}{\partial x_1^{j_1} \partial x_2^{j_2}} \right) \qquad (i = 1, 2) \tag{2.8}$$

whereby

$$f_{11}^{20} = -\frac{A_2 k_2^2}{A_1 (k_1^2 + k_2^2)}, \qquad f_{11}^{02} = -\frac{A_1}{2A_2}, \qquad f_{12}^{11} = f_{21}^{11} = -\frac{(k_1 - k_2)^2}{2(k_1^2 + k_2^2)}$$

$$f_{22}^{20} = -\frac{A_2}{2A_1}$$

$$f_{22}{}^{02} = -\frac{A_1 k_1{}^2}{A_2(k_1{}^2 + k_2{}^2)},\ f_{11}{}^{11} = f_{12}{}^{20} = f_{12}{}^{02} = f_{21}{}^{20} = f_{21}{}^{02} = f_{22}{}^{11} = 0$$

It is not difficult to calculate

$$\det\left(\sum_{j_1+j_2=2} f_{ij}^{j_1 j_2}\alpha_1{}^{j_1}\alpha_2{}^{j_2}\right)_{i,\ j=1,\ 2} = \frac{[(A_2/A_1)k_2\alpha_1{}^2 + (A_1/A_2)k_1\alpha_2{}^2]^2}{2(k_1{}^2 + k_2{}^2)}$$

Hence it follows that system (2.8) is a system of elliptic type in region G, since $k_1 k_2 > 0$ in that region.

As is known (see for example [3]), there exists a fundamental matrix

$$\omega(x, y) = \begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix}$$

of system (2.8). Each column of this matrix, with x G and $x \neq y$, satisfies the same system.

Consider the following system of vectors

$$\varphi_1 = \omega_{11}\mathbf{e}_1 + \omega_{21}\mathbf{e}_2 + \omega_{31}\mathbf{e}_3, \qquad \varphi_2 = \omega_{12}\mathbf{e}_1 + \omega_{22}\mathbf{e}_2 + \omega_{32}\mathbf{e}_3$$

where functions $\omega_{3j}(j = 1, 2)$ are determined by formula (2.7).

It is easily seen that for x G and $x \neq y$ the vectors $\phi_j(j = 1, 2)$ satisfy system (2.6).

Let the point $y(y_1, y_2)$ lie within G. We isolate this point by a circle K_ϵ of radius ϵ. We form the integral

$$\iint_{G-K_\varepsilon} \sum_{i,\ j=1}^{2} \varepsilon_{ij}(\mathbf{v})\,\varepsilon_{ij}(\varphi_n)\,A_1 A_2 dx_1 dx_2 \qquad (n = 1, 2)$$

From this integral, using equation (2.5), system (2.6) and boundary condition (2.2), and passing to the limit as $\epsilon \to 0$, we obtain

$$v_n(y) = \iint_G \sum_{i,\ j=1}^{2} \varepsilon_{ij}(\mathbf{v})\,\varepsilon_{ij}(\varphi_n)\,A_1 A_2 dx_1 dx_2 \qquad (n = 1, 2) \tag{2.9}$$

The kernels of the integrals possess a weak singularity, and these integrals, as is known, bounded in $L_2(G)$, are operators on $\epsilon_{ij}(\mathbf{v})$. From the boundedness of these operators it follows

$$\iint_G \sum_{n=1}^{2} v_n{}^2 dx_1 dx_2 \leqslant \gamma_1 \iint_G \sum_{i,\ j=1}^{2} \varepsilon_{ij}^{2}(\mathbf{v})\,dx_1 dx_2 \tag{2.10}$$

where γ_1 = const > 0 does not depend on $\mathbf{v}$.

Applying inequality $2ab \geqslant -(a^2+b^2)$, from (2.4) we can obtain

$$(\mathbf{Bv},\mathbf{v})+\frac{EA_0(1-\sigma)}{\sigma(1+\sigma)}\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}'')^2\,dx_1dx_2 \geqslant \frac{EA_0(1-\sigma)}{(1+\sigma)}\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}')^2\,dx_1dx_2 \tag{2.11}$$

$$\varepsilon_{11}'=\frac{1}{A_1}\frac{\partial v_1}{\partial x_1}-k_1v_3, \qquad \varepsilon_{11}''=\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}v_2$$

$$\varepsilon_{22}'=\frac{1}{A_2}\frac{\partial v_2}{\partial x_2}-k_2v_3, \qquad \varepsilon_{22}''=\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}v_1$$

$$\varepsilon_{12}'=\varepsilon_{21}'=\frac{1}{2}\left(\frac{1}{A_2}\frac{\partial v_1}{\partial x_2}+\frac{1}{A_1}\frac{\partial v_2}{\partial x_1}\right)$$

$$\varepsilon_{12}''=\varepsilon_{21}''=-\frac{1}{2A_1A_2}\left(\frac{\partial A_1}{\partial x_2}v_1+\frac{\partial A_2}{\partial x_1}v_2\right)$$

Now from (2.4), applying (2.10), we can deduce

$$(\mathbf{Bv},\mathbf{v})\geqslant \gamma_2\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}'')^2\,dx_1dx_2 \tag{2.12}$$

where γ_2 = const > 0 does not depend on v.

From (2.11), by virtue of (2.12), it follows

$$(\mathbf{Bv},\mathbf{v})\geqslant \gamma_3\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}')^2\,dx_1dx_2 \tag{2.13}$$

where γ_3 = const > 0 does not depend on $\mathbf{v}$.

It is easily deducible that

$$\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}')^2\,dx_1dx_2 \geqslant \gamma_4\iint_G [(k_2\varepsilon_{11}')^2+(k_1\varepsilon_{22}')^2+2(\varepsilon_{12}')^2]\,dx_1dx_2 \tag{2.14}$$

where γ_4 = const > 0 does not depend on $\mathbf{v}$.

From (2.14), applying the inequality $a^2+b^2 \geqslant 1/2(a-b)^2$ and the equation

$$\iint_G\left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2}\right)dx_1dx_2=0$$

we can deduce

$$\iint_G \sum_{i,j=1}^{2}(\varepsilon_{ij}')^2\,dx_1dx_2 \geqslant \gamma_5\iint_G\left[\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial v_2}{\partial x_2}\right)^2\right]dx_1dx_2 \tag{2.15}$$

where γ_5 = const > 0 does not depend on $\mathbf{v}$.

Now we can write down the following inequality:

$$\iint_G \sum_{i,j=1}^{2} \sum_{i,\ j=1}^{2} (\varepsilon_{ij}')^2\, dx_1 dx_2 \geqslant \iint_G [(\varepsilon_{11}{}^\circ - k_1 v_3)^2 + (\varepsilon_{22}{}^\circ - k_2 v_3)^2]\, dx_1 dx_2 \quad (2.16)$$

where

$$\varepsilon_{11}{}^\circ = \frac{1}{A_1}\frac{\partial v_1}{\partial x_1}, \qquad \varepsilon_{22}{}^\circ = \frac{1}{A_2}\frac{\partial v_2}{\partial x_2}$$

From (2.16), by virtue of $2ab \leqslant a^2 + b^2$, we obtain

$$\iint_G \sum_{i,j=1}^{2} (\varepsilon_{ij}')^2\, dx_1 dx_2 + \frac{1}{\sigma}(1-\sigma)\gamma_6 \iint_G \left[\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_2}{\partial x_2}\right)^2\right] dx_1 dx_2 \geqslant$$
$$\geqslant (1-\sigma)\gamma_7 \iint_G v_3{}^2\, dx_1 dx_2 \quad (2.17)$$

where

$$\frac{1}{A_1{}^2} \leqslant \gamma_6, \qquad \frac{1}{A_2{}^2} \leqslant \gamma_6, \qquad k_1{}^2 + k_2{}^2 \geqslant \gamma_7 > 0 \quad \text{in } G+L$$

Applying (2.15), from (2.17) we can now deduce

$$\iint_G \sum_{i,\ j=1}^{2} (\varepsilon_{ij}')^2 dx_1 dx_2 \geqslant \gamma_8 \iint_G v_3{}^2\, dx_1 dx_2 \quad (2.18)$$

where γ_8 = const > 0 does not depend on **v**.

From (2.13), by virtue of (2.18), it follows that

$$(\mathbf{Bv}, \mathbf{v}) \geqslant \gamma_9 \iint_G v_3{}^2\, dx_1 dx_2 \quad (2.19)$$

where γ_9 = const > 0 does not depend on **v**.

Now from (2.4), applying (2.10), we may also deduce

$$(\mathbf{Bv}, \mathbf{v}) \geqslant \frac{EA_0}{(1+\sigma)\gamma_1} \iint_G (v_1{}^2 + v_2{}^2)\, dx_1 dx_2 \quad (2.20)$$

Adding the inequalities (2.19) and (2.20), we obtain

$$(\mathbf{Bv}, \mathbf{v}) \geqslant \gamma_{10} \iint_G |\mathbf{v}|^2\, dx_1\, dx_2 \quad (2.21)$$

where γ_{10} = const > 0 does not depend on **v**.

From equation (2.1) we can deduce the following equality

$$(\mathbf{Bv}, \mathbf{v}) + h^2(\mathbf{Nv}, \mathbf{v}) = \mathbf{Qv} \quad (2.22)$$

Applying the boundary condition (2.2), we easily obtain

$$(\mathbf{N}\mathbf{v},\mathbf{v}) \geqslant \frac{E}{12(1+\sigma)} \sum_{i,j=1}^{2} \varkappa_{ij}^2(\mathbf{v}) A_1 A_2 dx_1 dx_2 \tag{2.23}$$

Taking into account (2.4) and (2.23), from (2.22) we can obtain

$$(\mathbf{Q}\mathbf{v}) \geqslant (\mathbf{B}\mathbf{v},\mathbf{v})$$

Hence, applying (2.21), there follows

$$(\mathbf{Q},\mathbf{v}) \geqslant \gamma_{10} \iint_G |\mathbf{v}|^2 dx_1 dx_2 \tag{2.24}$$

From (2.24), by virtue of $\mathbf{Q}\mathbf{v} \leqslant 1/2(|\mathbf{v}|^2 + |\mathbf{Q}|^2)$, we can deduce

$$\frac{1}{2\gamma_{10}} \iint_G |\mathbf{Q}|^2 dx_1 dx_2 + \frac{\gamma_{10}}{2} \iint_G |\mathbf{v}|^2 dx_1 dx_2 \geqslant \gamma_{10} \iint_G |\mathbf{v}|^2 dx_1 dx_2$$

Hence there follows the basic inequality (2.3), and the lemma is proved.

3. Proof of the basic theorem. We now proceed to the proof of the basic theorem formulated in Section 1.

Vector $\mathbf{U}_2' = \mathbf{U} - \mathbf{U}_0 = U_{12}\mathbf{e}_1 + U_{22}\mathbf{e}_2 + U_{32}\mathbf{e}_3$, by virtue of (1.6) and (1.8), satisfies the equation

$$\mathbf{B}\mathbf{U}_2' + h^2\mathbf{N}\mathbf{U}_2' = A_1 A_2 \mathbf{q}_1(x_1, x_2, h) - h^2 \mathbf{N}\mathbf{U}_0 \tag{3.1}$$

and by virtue of (1.7), (1.9), the boundary condition

$$U_{12}' = U_{22}' = 0, \qquad U_{32}' = -U_{30}, \ \frac{\partial U_{32}'}{\partial \nu} = -\frac{\partial U_{30}}{\partial \nu} \quad \text{on } L \tag{3.2}$$

We introduce the vector $\mathbf{U}_1^* = U_{11}^*\mathbf{e}_1 + U_{21}^*\mathbf{e}_2 + U_{31}\mathbf{e}_3$ which satisfies the equation

$$\mathbf{B}\mathbf{U}_1^* + h^2\mathbf{N}\mathbf{U}_1^* = 0 \tag{3.3}$$

and the boundary condition (3.2).

The vector $\mathbf{U}_1^*$ will be sought in the form

$$\mathbf{U}_1^* = \mathbf{U}_1 + R \tag{3.4}$$

where the vector $\mathbf{U}_1$ is determined by formulas (1.11).

Introducing (3.4) into equation (3.3), we obtain

$$B_1\mathbf{R} + h^2N_1\mathbf{R} = \frac{Eh^{-1/2}}{1-\sigma^2}\left\{\left[2\,\frac{A_2}{A_1}p_1^2 + (1-\sigma)\,\frac{A_1}{A_2}p_2^2\right]a_1 + \right.$$
$$\left. + (1+\sigma)\,p_1p_2a_2 + 2A_2\,(k_1+\sigma k_2)\,p_1a_3\right\}\exp(-gh^{-1/2})\sin(gh^{-1/2}) +$$
$$+\frac{E}{1-\sigma^2}\left\{\left[2\,\frac{A_2}{A_1}p_1^2 + (1-\sigma)\,\frac{A_1}{A_2}p_2^2\right]c_1 + (1+\sigma)\,p_2p_2c_2 + f_1^{(1)}(a_1, a_2, a_3, g)\right\}\times$$
$$\times\exp(-gh^{-1/2})\sin(gh^{-1/2}) - \frac{E}{1-\sigma^2}\left\{\left[2\,\frac{A_2}{A_1}p_1^2 + (1-\sigma)\,\frac{A_1}{A_2}p_2^2\right]b_1 + \right.$$

$$\left. + (1+\sigma)\,p_1p_2b_2 + f_2^{(1)}(a_1, a_2, a_3, b_3, g)\right\}\exp(-gh^{-1/2})\cos(gh^{-1/2}) + \frac{E}{2\,(1-\sigma^2)}\times$$
$$\times\left\{\left[2\,\frac{A_2}{A_1}p_1^2 + (1-\sigma)\,\frac{A_1}{A_2}p_2^2\right]d_1 + (1+\sigma)\,p_1p_2d_2 + 2A_2\,(k_1+\sigma k_2)\,p_1b_3\right\}\times$$
$$\times\exp(-gh^{-1/2}) + \sum_{i=1}^{6}\{h^{1/2\,i}\,[F_i^{(1)}\cos(g^{-1/2}) + \Phi_i^{(1)}\sin(gh^{-1/2}) + \chi_i^{(1)}]\}\times$$
$$\times\exp(-gh^{-1/2}) \tag{3.5}$$

$$B_2\mathbf{R} + h^2N_2\mathbf{R} = \frac{Eh^{-1/2}}{1-\sigma^2}\left\{(1+\sigma)\,p_1p_2a_1 + \left[2\,\frac{A_1}{A_2}p_2^2 + (1-\sigma)\,\frac{A_2}{A_1}p_1^2\right]a_2 + \right.$$
$$\left. + 2A_1\,(\sigma k_1 + k_2)\,p_2a_3\right\}\exp(-gh^{-1/2})\sin(gh^{-1/2}) + \frac{E}{1-\sigma^2}\left\{(1+\sigma)\,p_1p_2c_1 + \right.$$
$$\left. + \left[2\,\frac{A_1}{A_2}p_2^2 + (1-\sigma)\,\frac{A_2}{A_1}p_1^2\right]c_2 + f_1^{(2)}(a_1, a_2, a_3, g)\right\}\exp(-gh^{-1/2})\sin(gh^{-1/2}) -$$
$$-\frac{E}{1-\sigma^2}\left\{(1+\sigma)\,p_1p_2b_1 + \left[2\,\frac{A_1}{A_2}p_2^2 + (1-\sigma)\,\frac{A_2}{A_1}p_1^2\right]b_2 + f_2^{(2)}(a_1, a_2, a_3, b_3, g)\right\}\times$$
$$\times\exp(-gh^{1/2})\cos(gh^{-1/2}) + \frac{E}{2\,(1-\sigma^2)}\left\{(1+\sigma)\,p_1p_2d_1 + \right.$$
$$\left. + \left[2\,\frac{A_1}{A_2}p_2^2 + (1-\sigma)\,\frac{A_2}{A_1}p_1^2\right]d_2 + 2A_1\,(\sigma k_1 + k_2)\,p_2b_3\right\}\exp(-gh^{-1/2}) +$$
$$+\sum_{i=1}^{6}\{h^{1/2\,i}\,[F_i^{(2)}\cos(gh^{-1/2}) + \Phi_i^{(2)}\sin(gh^{-1/2}) + \chi_i^{(2)}]\}\exp(-gh^{-1/2}) \tag{3.6}$$

$$B_3\mathbf{R} + h^2N_3\mathbf{R} = \frac{E}{1-\sigma^2}\left\{-A_2\,(k_1+\sigma k_2)\,p_1a_1 - A_1\,(\sigma k_1 + k_2)\,p_2a_2 + \right.$$
$$\left. + \left[-A_1A_2\,(k_1^2 + k_2^2 + 2\sigma k_1k_2) + \frac{1}{3A_1A_2}\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^2\right]a_3\right\}\times$$
$$\times[\sin(gh^{-1/2}) + \cos(gh^{-1/2})]\exp(-gh^{-1/2}) +$$
$$+\sum_{i=1}^{6}\{h^{1/2\,i}\,[F_i^{(3)}\cos(gh^{-1/2}) + \Phi_i^{(3)}\sin(gh^{-1/2}) + \chi_i^{(3)}]\}\exp(-gh^{-1/2}) \tag{3.7}$$

where

$$p_1 = \frac{\partial g}{\partial x_1}, \qquad p_2 = \frac{\partial g}{\partial x_2}$$

We consider the system of equations

$$\begin{gathered}
\left[2\frac{A_2}{A_1}p_1^2 + (1-\sigma)\frac{A_1}{A_2}p_2^2\right]a_1 + (1+\sigma)p_1p_2a_2 + 2A_2(k_1+\sigma k_2)p_1a_3 = 0 \\
(1+\sigma)p_1p_2a_1 + \left[2\frac{A_1}{A_2}p_2^2 + (1-\sigma)\frac{A_2}{A_1}p_1^2\right]a_2 + 2A_1(\sigma k_1 + k_2)p_2a_3 = 0 \qquad (3.8) \\
-A_2(k_1+\sigma k_2)p_1a_1 - A_1(\sigma k_1 + k_2)p_2a_2 + \\
+\left[-A_1A_2(k_1^2+k_2^2+2\sigma k_1k_2) + \frac{1}{3A_1A_2}\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^2\right]a_3 = 0
\end{gathered}$$

Equating to zero the determinant of this system, we obtain

$$\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^4 - 3(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)^2 = 0 \qquad (3.9)$$

We determine the function g such that it satisfies equation (3.9), that it vanishes on L and that it is positive at points of the region G. To prove the possibility of constructing such a function we apply the method presented in paper [4].

The differential equations of characteristics (3.9) have the following form (see for example [5], Section 56):

$$\frac{dx_1}{dt} = 8\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^3\frac{A_2}{A_1}p_1 - 12(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)A_2^2k_2p_1 \quad (3.10)$$

$$\frac{dx_2}{dt} = 8\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^3\frac{A_1}{A_2}p_2 - 12(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)A_1^2k_1p_2 \quad (3.11)$$

$$\begin{gathered}
\frac{dp_1}{dt} = -4\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^3\left[p_1^2\frac{\partial}{\partial x_1}\left(\frac{A_2}{A_1}\right) + p_2^2\frac{\partial}{\partial x_1}\left(\frac{A_1}{A_2}\right)\right] + \\
+6(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)\left[p_1^2\frac{\partial}{\partial x_1}(A_2^2k_2) + p_2^2\frac{\partial}{\partial x_1}(A_1^2k_1)\right] \qquad (3.12)
\end{gathered}$$

$$\begin{gathered}
\frac{dp_2}{dt} = -4\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^3\left[p_1^2\frac{\partial}{\partial x_2}\left(\frac{A_2}{A_1}\right) + p_2^2\frac{\partial}{\partial x_2}\left(\frac{A_1}{A_2}\right)\right] + \\
+6(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)\left[p_1^2\frac{2\partial}{\partial x_2}(A_2^2k_2) + p_2^2\frac{\partial}{\partial x_2}(A_1^2k_1)\right]
\end{gathered} \qquad (3.13)$$

$$\frac{dg}{dt} = 8\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^4 - 12(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)^2 \qquad (3.14)$$

Let the boundary L be given by equations $x_1 = x_1(s)$, $x_2 = x_2(s)$, where s is the arc length of the curve L.

Applying (3.10) and (3.11) we obtain

$$x_1'(s)\frac{dx_2}{dt} - x_2'(s)\frac{dx_1}{dt} = 8\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^3\left[\frac{A_1}{A_2}x_1'(s)p_2 - \frac{A_2}{A_1}x_2'(s)p_1\right] - $$
$$- 12(1-\sigma^2)(A_2^2k_2p_1^2 + A_1^2k_1p_2^2)[A_1^2k_1x_1'(s)p_2 - A_2^2k_2x_2'(s)p_1] \quad (3.15)$$

Since $g = 0$ on L, it follows

$$p_1x_1'(s) + p_2x_2'(s) = 0$$

By virtue of the last of equations (3.9) we may obtain

$$p_1 = -\sqrt[4]{3(1-\sigma^2)}\frac{[A_2^2k_2x_2'^2(s) + A_1^2k_1x_1'^2(s)]^{1/2}}{(A_2/A_1)x_2'^2(s) + (A_1/A_2)x_1'^2(s)}x_2'(s)$$
$$p_2 = \sqrt[4]{3(1-\sigma^2)}\frac{[A_2^2k_2x_2'^2(s) + A_1^2k_1x_1'^2(s)]^{1/2}}{(A_2/A_1)x_1'^2(s) + (A_1/A_2)x_2'^2(s)}x_1'(s) \quad (3.16)$$

Introducing (3.16) into (3.15), we obtain

$$x_1'(s)\frac{dx_2}{dt} - x_2'(s)\frac{dx_1}{dt} =$$
$$= 12(1-\sigma^2)\sqrt[4]{3^3(1-\sigma^2)^3}\frac{[A_2^2k_2x_2'^2(s) + A_1^2k_1x_1'^2(s)]^{7/2}}{[(A_2/A_1)x_2'^2(s) + (A_1/A_2)x_1'^2(s)]^3} \neq 0$$

Thus, at each point of the boundary L the projection on the plane $x_1 + ix_2$ of the characteristic does not touch L.

As is known (see for example [5], Section 56), it follows from this that there exists a function g in a certain neighborhood of the boundary L which satisfies the equation (3.9) and vanishes on L.

Receding from the boundary L (where $g = 0$) into G along the characteristics, from (3.14), by virtue of (3.9), we may deduce

$$\frac{dg}{dt} = 4\left(\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2\right)^4 \quad (3.17)$$

By virtue of equations (3.16) we will have

$$\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2 > 0 \qquad \text{on } L$$

Hence we may conclude that owing to continuity there exists a certain small vicinity Ω of the boundary L, in which

$$\frac{A_2}{A_1}p_1^2 + \frac{A_1}{A_2}p_2^2 > 0 \quad (3.18)$$

From (3.17), applying the inequality (3.18), we may deduce that in the region Ω $dg/dt = 0$. It follows from this that $g > 0$ at points of the region G belonging to the vicinity of Ω of the boundary L.

We note that in the case of the spherical segment $0 \leqslant \theta \leqslant \theta_0$, the function g has the form:

$$g = r^{\frac{1}{2}} \sqrt[4]{3(1-\sigma^2)}\,(\vartheta_0 - \vartheta)$$

where r is the radius of the sphere. In this case the width of the boundary layer δ, measured in radians, may be calculated by the formula

$$\delta = \frac{4}{\sqrt[4]{3(1-\sigma^2)}} \left(\frac{h}{r}\right)^{1/2}$$

The following equations hold good on L because of the properties of the function g and the boundary condition (3.2):

$$\begin{aligned} &a_1 h^{1/2} + h(c_1 + d_1) + R_1 = 0 \\ &a_2 h^{1/2} + h(c_2 + d_2) + R_2 = 0 \\ &a_3 + R_3 = -U_{30}, \qquad \frac{\partial a_3}{\partial \nu} + b_3 \frac{\partial g}{\partial \nu} + \frac{\partial R_3}{\partial \nu} = -\frac{\partial U_{30}}{\partial \nu} \end{aligned} \tag{3.19}$$

As functions a_3, b_3 we may take arbitrary functions which are continuously differentiable a sufficient number of times, which satisfy the boundary condition

$$a_3 = -U_{30}, \quad b_3 = -\left(\frac{\partial U_{30}}{\partial \nu} + \frac{\partial a_3}{\partial \nu}\right) \Big/ \frac{\partial g}{\partial \nu} \qquad \text{on } L \tag{3.20}$$

and which vanish outside Ω. It is not difficult to show that $\partial g / \partial \nu > 0$ on L.

The functions a_1, a_2 are determined from the following system:

$$\begin{aligned} &\left[2\frac{A_2}{A_1} p_1^2 + (1-\sigma)\frac{A_1}{A_2} p_2^2\right] a_1 + (1+\sigma) p_1 p_2 a_2 = -2A_2(k_1 + \sigma k_2) p_1 a_3 \\ &(1+\sigma) p_1 p_2 a_1 + \left[2\frac{A_1}{A_2} p_2^2 + (1-\sigma)\frac{A_2}{A_1} p_1^2\right] a_2 = -2A_1(\sigma k_1 + k_2) p_2 a_3 \end{aligned} \tag{3.21}$$

The determinant of this system has the form:

$$2(1-\sigma)\left(\frac{A_2}{A_1} p_1^2 + \frac{A_1}{A_2} p_2^2\right)^2$$

It is seen that, by virtue of (3.18), the determinant is different from zero and therefore the system (3.21) is solvable in Ω. Outside Ω the functions a_1, a_2 are equated to zero.

In this fashion the functions a_1, a_2, a_3 satisfy the system (3.8), since the determinant of this homogeneous system is equal to zero.

The functions b_1, b_2, c_1, c_2, d_1, d_2 are determined from the following systems:

$$\left[2\frac{A_2}{A_1}p_1^2+(1-\sigma)\frac{A_1}{A_2}p_2^2\right]c_1+(1+\sigma)p_1p_2c_2=-f_1^{(1)}(a_1,\ a_2,\ a_3,\ g) \quad (3.22)$$

$$(1+\sigma)p_1p_2c_1+\left[2\frac{A_1}{A_2}p_2^2+(1-\sigma)\frac{A_2}{A_1}p_1^2\right]c_2=-f_1^{(2)}(a_1,\ a_2,\ a_3,\ g)$$

$$\left[2\frac{A_2}{A_1}p_1^2+(1-\sigma)\frac{A_1}{A_2}p_2^2\right]b_1+(1+\sigma)p_1p_2b_2=-f_2^{(1)}(a_1,\ a_2,\ a_3,\ b_3,\ g)$$

$$(3.23)$$

$$(1+\sigma)p_1p_2b_1+\left[2\frac{A_1}{A_2}p_2^2+(1-\sigma)\frac{A_2}{A_1}p_1^2\right]b_2=-f_2^2(a_1,\ a_2,\ a_3,\ b_3,\ g)$$

$$\left[2\frac{A_2}{A_1}p_1^2+(1-\sigma)\frac{A_1}{A_2}p_2^2\right]d_1+(1+\sigma)p_1p_2d_2=-2A_2(k_1+\sigma k_2)p_1b_3$$

$$(3.24)$$

$$(1+\sigma)p_1p_2d_1+\left[2\frac{A_1}{A_2}p_2^2+(1-\sigma)\frac{A_2}{A_1}p_1^2\right]d_2=-2A_1(\sigma k_1+k_2)p_2b_3$$

We note that the right-hand sides of these systems vanish, together with a_1, a_2, a_3, b_3, and therefore the functions c_1, c_2, b_1, b_2, d_1, d_2 may be equated to zero outside Ω.

The vector $\mathbf{R}$, by virtue of (3.8), (3.22), (3,23), (3.24), (3.5), (3.6), (3.7), satisfies the equation

$$\mathbf{BR}+h^2\mathbf{NR}=\sum_{i=1}^{6}\{h^{1/2 i}[\mathbf{F}_i\cos(gh^{-1/2})+\mathbf{\Phi}_i\sin(gh^{-1/2})+\chi_i]\}\exp(-gh^{-1/2}) \quad (3.25)$$

and by virtue of (3.19), (3.20), the boundary condition

$$a_1h^{1/2}+h(c_1+d_1)+R_1=0,\qquad a_2h^{1/2}+h(c_2+d_2)+R_2=0$$

$$R_3=\frac{\partial R_3}{\partial \nu}=0\qquad \text{on } L \quad (3.26)$$

Let us consider the vector $\mathbf{R}'=h^{1/2}\mathbf{r}_1+h\mathbf{r}_2+\mathbf{R}=R_1'\mathbf{e}_1+R_2'\mathbf{e}_2+R_3'\mathbf{e}_3$ where

$$\mathbf{r}_1=a_1\mathbf{e}_1+a_2\mathbf{e}_2,\qquad \mathbf{r}_2=(c_1+d_1)\mathbf{e}_1+(c_2+d_2)\mathbf{e}_2$$

This vector, by virtue of (3.25), satisfies the equation

$$\mathbf{BR}'+h^2\mathbf{NR}'=\sum_{i=1}^{6}\{h^{1/2 i}[\mathbf{F}_i\cos(gh^{-1/2})+\mathbf{\Phi}_i\sin(gh^{-1/2})+$$

$$+\chi_i]\}\exp(-gh^{-1/2})+h^{1/2}\mathbf{Br}_1+h\mathbf{Br}_2+h^{5/2}\mathbf{Nr}_1+h^3\mathbf{Nr}_2 \quad (3.27)$$

and, by virtue of (3.26), the boundary condition

$$R_1' = R_2' = R_3' = \frac{\partial R_3'}{\partial \nu} = 0 \quad \text{on } L \tag{3.28}$$

The basic inequality (2.3) may be applied to the vector $\mathbf{R}$, since, by virtue of (3.28), it satisfies the conditions of the lemma, and therefore, taking into account the right-hand side of equation (3.27), we obtain

$$\lim_{h\to 0} \iint_G |\mathbf{R}'|^2 \, dx_1 \, dx_2 = 0$$

It follows that

$$\lim_{h\to 0} \iint_G |\mathbf{R}|^2 \, dx_1 \, dx_2 = 0 \tag{3.29}$$

We now consider the vector

$$\mathbf{U}_2^* = \mathbf{U}_2' - \mathbf{U}_1^* = U_{12}^* \mathbf{e}_1 + U_{22}^* \mathbf{e}_2 + U_{32}^* \mathbf{e}_3$$

which, by virtue of (3.1), (3.3), satisfies the equation

$$\mathbf{B}\mathbf{U}_2^* + h^2 \mathbf{N}\mathbf{U}_2^* = A_1 A_2 \mathbf{q}_1 (x_1, \ x_2, \ h) - h^2 \mathbf{N}\mathbf{U}_0 \tag{3.30}$$

and, by virtue of (3.2), the boundary condition (3.28).

Now applying to the vector $\mathbf{U}_2^*$ the basic inequality (2.3), since it satisfies the conditions of the lemma, and taking into account the right-hand side of equation (3.30) and condition (1.5), we obtain

$$\lim_{h\to 0} \iint_G |\mathbf{U}_2^*|^2 \, dx_1 \, dx_2 = 0 \tag{3.31}$$

It is not difficult to derive

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1^* + \mathbf{U}_2^*$$

Hence, applying formula (3.4), we obtain the inequality (1.10), where

$$\mathbf{U}_2 = \mathbf{R} + \mathbf{U}_2^* \tag{3.32}$$

Now, taking into account (3.29), (3.31) and (3.32), we obtain condition (1.12). The basic theorem is thus proved.

It is not difficult to prove that if the condition

$$\iint_G |\mathbf{q}_1|^2 \, dx_1 \, dx_2 = 0\,(h)$$

is satisfied, then

$$\iint_G |\mathbf{U}_2|^2 \, dx_1 \, dx_2 = 0\,(h).$$

It follows from the basic theorem that

$$\lim_{h\to 0} \iint_G |\mathbf{U} - \mathbf{U}_0 - \mathbf{U}_1|^2 \, dx_1 \, dx_2 = 0$$

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Translated by G.H.